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FINITE [Q-OSCILLATOR] REPRESENTATION OF 2-D STRING THEORY

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Abstract

We present a simple physical representation for states of two-dimensional string theory. In order to incorporate a fundamental cutoff of the order $1/g_{\rm st}$ we use a picture consisting of q-oscillators at the first quantized level. In this framework we also find a representation for the (singular) negatively dressed states representing nontrivial string backgrounds.

1. Introduction

Two-dimensional string theory [1,2] possesses a most interesting structure. Its spectrum contains, in addition to a massless scalar particle (the tachyon) [3,4], also an infinite sequence of discrete states with a closely related W_∞ symmetry algebra [5,6]. In the world sheet picture with $X_\mu = (X, \varphi)$, this spectrum is given by the vertex operators

$$\Psi^{(\pm)}_{Jm} \equiv \left(H_-(z)\right)^{J-m} : e^{iJX(z)} : e^{(-1\pm J)\varphi(z)} : \tag{1.1}$$

with discrete momenta $p_X = m$, $p_\varphi = \pm J$.

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Of some importance is the difference between the states of positive Liouville dressing $\Psi^{(+)}$ versus those of negative dressing $\Psi^{(-)}$. The former are physical states. In particular, for tachyons they provide left ($p_X > 0$) and right ($p_X < 0$) moving scattering states. The negatively dressed states to which we shall refer as "singular" violate the so-called Seiberg bound. They do not have a simple scattering interpretation. They do play an important role, however, in providing nontrivial backgrounds, among which the first is the two dimensional black hole [7].

The difference between the two sets of states is clearly seen in their operator product algebras [8]:

$$
\begin{aligned}
\Psi^{(+)}_{J_1 m_1}(z)\Psi^{(+)}_{J_2 m_2}(w) &= \frac{1}{z-w}(J_2 m_1 - J_1 m_2)\Psi^{(+)}_{J_1+J_2-1, m_1+m_2}(w) \\
\Psi^{(+)}_{J_1 m_1}(z)\Psi^{(-)}_{J_2, m_2}(w) &= \frac{1}{z-w}((J_2+1)m_1 - J_1 m_2)\Psi^{(-)}_{J_2-J_1+1, m_1+m_2}(w), \\
&\qquad\qquad J_1 < J_2 + 1 \\
&\qquad |m_1+m_2| \le J_2 - J_1 + 1 \\
\Psi^{(+)}_{J_1 m_1}(z)\Psi^{(-)}_{J_2 m_2}(w) &= 0, \quad \text{otherwise}, \\
\Psi^{(-)}_{J_1 m_1}(z)\Psi^{(-)}_{J_2 m_2}(w) &= 0.
\end{aligned}
\tag{1.2}
$$

Here the positively dressed vertex operators close a W_∞ algebra while the operator product among negatively dressed states is trivial.

Matrix models seem to capture very well certain aspects of the theory. In particular, the physical (positively dressed) operators appear as

$$
Tr\left((P+M)^{J+m}(P-M)^{J-m}\right), \tag{1.3}
$$

or

$$
\int dx\, \psi^\dagger(x)(a^\dagger)^{J+m} a^{J-m}\psi(x) \tag{1.4}
$$

in terms of free fermions. In this formulation we can very simply solve the scattering problem to arbitrary order [9]. This represents a notable success of the matrix model approach.

In the matrix model, however, the presence or interpretation of the negatively dressed (singular) states is at best questionable. For example, a naive extension $J \to -J$ in (1.4) would lead to singular-seeming expressions of the type $(a^\dagger)^{-\bar{n}}a^{-n}$. Apart from the mathematical difficulties involved in trying to make sense of such operators, it is hard to see how they could be made to satisfy anything close to the algebra given above in (1.2).

A second basic problem seems to appear in the transition to the (collective) field theory representation. There the states are represented in terms of a massless scalar field, in particular

$$\int dx \int d\alpha \, (\alpha + x)^{J+m} (\alpha - x)^{J-m},$$

where $\alpha_{\pm} = \pi_{,x} \pm \phi$. The string coupling constant $g_{\rm st} = 1/N$ now appears through the constraint on the total number of nonrelativistic particles $\int dx \, \phi = N$. Consequently, the ground state is given by the filled fermi vacuum, with excitations being particles and holes respectively above and below the fermi surface. Perturbation theory explores the region close to the Fermi surface. Any description of large excitations, however, has to take into account that holes cannot be deeper than N, and we expect serious modifications of the theory for momenta close to $1/g_{\rm st}$. This issue is of central relevance in connection with the evaluation of nonperturbative effects, which in string theory are expected to be of order $e^{-1/g_{\rm st}}$.

A proper understanding and satisfactory description of the singular states, as well as the nonperturbative cutoff $1/g_{\rm st}$, are therefore of basic interest.

In what follows we shall address these questions. We shall provide a simple physical picture in which both the singular states and the cutoff will appear naturally. In our opinion the two issues are closely related. In the representation that we outline a central role will be played by quantum oscillators with deformation parameter a root of unity. This introduces a cutoff of order $1/g_{\rm st}$ and in addition provides a simple framework for the interpretation of the singular (negatively dressed) states. As we shall argue, these will correspond to excitations far below the fermi surface. A picture based on a q-oscillator representation is then offered as a simple framework in which to study the interaction between excitations near the Fermi surface and those which are very deep below it.

2. From conformal field theory to free fermions

The correspondence between the world sheet, conformal field theory and the target space or matrix model description of string theory is at present known only on a rather heuristic level. For this reason we begin with a reexamination of what seems to be the best understood subject: the positively dressed states. As we have seen, analogous objects in the matrix model are given by (1.3) or (1.4). However, the full physical meaning of these fermionic states has not yet been clarified.

A clue for the interpretation that follows and much of our analysis comes from realizing the following fact: The continuum vertex operator representation (1.1) displays a simple exponential dependence $e^{p_\varphi \phi}$ on the Liouville mode. This is suggestive

of a hamiltonian description in which φ_0 (the zero mode of $\varphi(z)$) plays the role of time. In the matrix model, however, the hamiltonian time is a priori the target space coordinate, which would be expected to correspond to the zero mode of the conformal field coordinate $X(z)$.

This mismatch of space and time coordinates between the two approaches seems to imply a serious obstacle to any comparison between them. However, as we shall indicate below, in the matrix model an interchange of the roles of space and time, at least in an analytically continued sense, at most changes the boundary conditions in its fermionic description. For the moment assuming this, we shall proceed by looking for an identification between conformal field theory states and fermionic (matrix model) excitations, associating the zero mode of φ to the matrix-model time coordinate (and therefore the matrix model energy levels to the eigenvalues J of the conjugate variable $\hat{p}_\varphi$).

Consider now in more detail the precise form in terms of oscillators of the conformal field theory states. For example, for the sequence of left-moving states of the form

$$\Psi^{(+)}_{J,J-1} \equiv H_-(z) : e^{iJX(z)} : e^{(-1+J)\varphi(z)} : \tag{2.1}$$

in the hilbert space with oscillators introduced as

$$\partial X(z) = \sum_n \alpha_n z^{-n-1},$$

where

$$\alpha_n = a_n \quad \text{and} \quad \alpha_{-n} = n a_n^\dagger,$$

we find

$$\left| \Psi^{(+)}_{J,J-1} \right\rangle = S_{2J-1}(a_n) \left| 0 \right\rangle e^{\sqrt{2}(-1+J)\varphi_0},$$

where the S_n are Schur polynomials of order n.

Similarly, for the right-moving states we find

$$\left| \Psi^{(+)}_{J,-J+1} \right\rangle = S_{2J-1}(-a_n) \left| 0 \right\rangle e^{\sqrt{2}(-1+J)\varphi_0}.$$

The Schur polynomials are known to have a physical interpretation as excitations in a free fermion theory. In particular, $S_n(a)$, corresponding to the young tableau with one row and n columns, excites a fermion at n steps above the fermi surface. Similarly, the second series of states corresponds to hole excitations. Thus, we see that the vertex operator representation translates into a nonrelativistic fermion representation with the Liouville mode playing the role of time. As we have discussed, this fermionic picture of the vertex operator states is dual to the standard matrix model (fermion) picture in the sense that the roles of time and space are interchanged.

In general, it is a fermionic interpretation that brings in the requirement of a cutoff; namely, it is known that the sequence of polynomials that corresponds to the column states must terminate once the column is longer than N; in other words, a hole cannot be deeper than the fermi momentum. This constraint is not obviously recognizable in conformal field theory itself. Its implementation will be our first concern.

Before presenting the main discussion, let us close the section with the promised comment on how the matrix model transforms under an interchange of space and time directions.

Taking the free fermion representation that arises in the matrix model, and assuming periodicity in the holomorphic representation in terms of the analytically continued $z = x + ip$, we have, taking $H = \frac{1}{2}(z\partial + \partial z) = z\partial + 1/2$, the expansion

$$\psi = \sum_{n \in Z+1/2} \psi_n z^{n-1/2} e^{int}. \tag{2.2}$$

Interchanging the roles of energy and momentum by taking $z \leftrightarrow e^{it}$, we find

$$\psi \to \sum_{n \in Z+1/2} \psi_n z^n e^{i(n-1/2)t} \tag{2.3}$$

We now have Ramond fermions up to a renaming $\psi_{n-1/2} \to \psi_n$ of the modes. Thus, we see that in the free fermion description of the matrix model, interchanging the time and space directions as above will take Neveu-Schartz (periodic) fermions into Ramond (antiperiodic) fermions (this is also consistent with the fact that the plane to cylinder mapping ($z \to e^{it}$) changes the periodicity of the fermions [10]).

We remark that, by extension of the above argument, introducing a nonzero chemical potential μ in the matrix model Hamiltonian would translate in the rotated system into having twisted fermions with the phase of the twist depending on μ. In this paper the dependence on the number of fermions N will be taken into account explicitly through the use of q-oscillators and we will not need a nonzero μ. Allowing μ to vary would indeed correspond to allowing an additional degree of freedom, over and above those that are a priori provided by the matrix model.

3. Quantum oscillators and the cutoff

As we have seen, the fermionic interpretation of the conformal field theory states very naturally leads to the suggestion that there should be an explicit cutoff present on the space of states themselves, a constraint which is not a priori visible in the conformal field theory itself in its usual formulation. In particular, the holes should not be deeper than N (the total number of fermions, or order of the matrix), which is $1/g_{\rm st}$ in terms of the string coupling. If we want left- and right-moving states to appear symmetrically in the theory, we are motivated to impose the same cutoff on the momenta of the particle states.

In order to ensure that there arise no excitations deeper than N, we are led to suggest a representation in terms of q-oscillators, which we motivate as follows:

Consider a hole in a first quantized formalism, and denote the corresponding creation and annihilation operators by $a^\dagger$ and a. The fact that a hole cannot be excited deeper than N will be imposed by requiring

$$(a^\dagger)^N = a^N = 0.$$

This imposes the need to modify the standard commutation relations. One obvious candidate would be

$$[a, a^\dagger] = 1 - (\hat{N}+1)\delta_{\hat{N},N-1}, \tag{3.1}$$

where $\hat{N} = a^\dagger a$ is the number operator. The algebra of operators becomes finite-dimensional, with a basis given by the N^2 generators $(a^\dagger)^m a^n$, $m, n = 0, ..., N-1$. However, due to the second term on the right hand side in (3.1), it is not easy to see how to define a consistent coproduct [11], which will be needed to extend the algebra to a multi-particle representation in the second quantized theory. However, we can already see possible candidates for the singular states as being given by the ones near the top of the "stack", namely $(a^\dagger)^{N-n}a^{N-m}$.

One solution to the problem of the coproduct is given in terms of q-deformed oscillators satisfying

$$\begin{aligned} bb^\dagger - q^2 b^\dagger b &= 1, \\ [\hat{N}, b^\dagger] = [b, \hat{N}] &= 1, \end{aligned} \tag{3.2}$$

where q is a root of unity, satifying $q^{2N} = 1$, $\hat{N}$ is the number operator and $b\,|0\rangle = \hat{N}\,|0\rangle = 0$. It can be shown that in this representation we have $b^N = (b^\dagger)^N = 0$, which incorporates the cutoff. It is easy to convince ourselves that we can express

such oscillators in terms of the a's of (3.1) above by defining, for example,

$$
\begin{gathered}
b^\dagger = \sqrt{\frac{[\hat{N}]}{\hat{N}}}a^\dagger, \qquad b = a\sqrt{\frac{[\hat{N}]}{\hat{N}}}, \\
\hat{N} \to \hat{N}, \\
[p] \equiv \frac{q^{2p-1}}{q^2-1}.
\end{gathered}
$$

For q a phase, as it will be in our case, $b^\dagger$ will no longer be the hermitian conjugate of b. A related set of oscillators with simpler hermiticity properties is given by $a = q^{-\hat{N}/2}b$, $a^\dagger = b^\dagger q^{-\hat{N}/2}$, where now $(a^\dagger)^\dagger = a$ and we get the q-commutation relations

$$
\begin{aligned}
aa^\dagger - qa^\dagger a &= q^{-\hat{N}}, \\
aa^\dagger - q^{-1}a^\dagger a &= q^{\hat{N}}.
\end{aligned}
\tag{3.3}
$$

For a review, see [12,13]. Discussion of the coproduct will be postponed to section 4.

Define as a basis for the operators on the one-hole hilbert space

$$
O_{Jm} \equiv (b^\dagger)^{J+m} b^{J-m}. \tag{3.4}
$$

These operators satisfy a deformed version of a W-algebra: Following Zha [11], we define a deformed version of the commutator, which we shall call a q-bracket, as

$$
\begin{aligned}
[O_{Jm}, O_{J'm'}]_q &\equiv q^{-2(J'm-Jm')}\, O_{Jm}\, O_{J'm'} - q^{2(J'm-Jm')}\, O_{J'm'}\, O_{Jm} \\
&= ([J'+m']_q\,[J-m]_q\, q^{-2m} - [J+m]_q\,[J'-m']_q\, q^{-2m'})\, O_{J+J'-1,m+m'},
\end{aligned}
\tag{3.5}
$$

up to terms proportional to $O_{J+J'-s}$ with $s \geq 2$, and where

$$
[x]_q \equiv \frac{q^N - q^{-N}}{q-q^{-1}}.
$$

The motivation for the powers of q appearing in the definition of the q-bracket is that they are what is needed for the term proportional to $O_{J+J',m+m'}$ to vanish on the right hand side. For the coefficients of all the lower order terms we refer the reader to [11].

The undeformed W-algebra is obtained as a limit of the q-deformed version above as $q \to 1$. In particular, in our case $q = e^{\pi i/N}$, where N is the cutoff, so that when

N becomes large and $x << N$, we get $[x]_q \to x$, while the q-bracket reduces to the ordinary commutator, so that for operators with sufficiently low J-values

$$\begin{aligned}[O_{Jm}, O_{J'm'}] &\to ((J'+m')(J-m) - (J+m)(J'-m'))O_{J+J'-1,m+m'} \\ &= 2(m'J - mJ')O_{J+J'-1,m+m'}\end{aligned}$$

As a bonus, in this picture we now get candidates for the negatively dressed (singular) states with the correct commutation relations (1.2) in the limit. Indeed, defining

$$\bar{O}_{Jm} \equiv O_{N-J-1,m}, \qquad |m| \le J$$

we find, using $q^{2N} = 1$, that for large N the q-bracket between O's and $\bar{O}$'s, and indeed between two $\bar{O}$'s, reduces to the ordinary commutator. Furthermore, from $[x]_q = [N-x]_q \to x$, it follows that

$$\begin{aligned}[O_{Jm}, \bar{O}_{J'm'}]_q = ([N - J' - 1 + m']_q[J-m]_q \\ - [J+m]_q[N - J' - 1 - m']_q)\bar{O}_{-J+J'+1,m+m'} \\ \to -2((J'+1)m + Jm')\bar{O}_{-J+J'+1,m+m'},\end{aligned} \tag{3.6}$$

as long as $J < J' + 1$ and $|m + m'| \le J' - J + 1$. If not, the result will be 0, because in the algebra (3.4) there are no operators with power of b or of $b^\dagger$ larger than $N - 1$. By the same token,

$$[\bar{O}_{Jm}, \bar{O}_{J'm'}] = 0. \tag{3.7}$$

With this we have demonstated that the algebraic structure of 2-d string theory vertex operators is completely reproduced in the q-oscillator phase space. This is particularly nontrivial for the singular (negatively dressed) states.

The algebra (3.6) and (3.7) is the same as the one (1.2) obtained in the conformal field theory approach. However, so far it only contains the hole contribution, and in the full theory we have to include the particles.

To do this, we introduce another set of q-creation and annihilation operators $\bar{b}^\dagger$ and $\bar{b}$ associated to the particles. In the following, we will use the notation

$$\begin{aligned}\partial, z \equiv b, b^\dagger, \\ \bar{\partial}, \bar{z} \equiv \bar{b}, \bar{b}^\dagger\end{aligned} \tag{3.8}$$

suggestive of a q-deformed holomorphic-antiholomorphic representation. Following [12], we extend the commutation relations to the z, $\bar{z}$ system by requiring associativity

of the differentiation rules such that the following braiding relations are satisfied:

$$\begin{aligned}&z\bar{z}=q^2\bar{z}z\\ &\partial\bar{\partial}=q^2\bar{\partial}\partial,\quad \partial\bar{z}=q^{-2}\bar{z}\partial,\quad \bar{\partial}z=q^2z\bar{\partial}\\ &\partial z=1+q^{-2}z\partial,\quad \bar{\partial}\bar{z}=1+q^2\bar{z}\bar{\partial}.\end{aligned}\tag{3.9}$$

To generalize the expression for the W-charge (3.4) to two oscillators, one might try the expression $(\partial+\frac{1}{2}\bar{z})^{J+m}(\bar{\partial}-\frac{1}{2}z)^{J-m}$. This works in the undeformed case $q=1$. However, in the deformed case $(\partial+\frac{1}{2}\bar{z})(\bar{\partial}-\frac{1}{2}z)-q^2(\partial\bar{z}-\frac{1}{2}z)(\partial z+\frac{1}{2}\bar{z})\neq 1$. This happens because ∂ and $\bar{\partial}$ pick up the "wrong" power of q when the are interchanged, and thus the terms in $\partial\bar{\partial}$ do not cancel. This problem can be avoided by taking as generators the expressions $(\bar{\partial}+\frac{1}{2}z)$ and $\bar{z}$, which do satisfy the q-commutation relation (3.2). We are therefore motivated to consider the following candidate for a particle-hole q-deformed W-algebra:

$$O_{Jm}=(\bar{\partial}+\frac{1}{2}z)^{J+m}\bar{z}^{J-m}.\tag{3.10}$$

These generators indeed satisfy the above q-deformed W-algebra (3.5).

The representation (3.10) suggests the following correspondence with the conformal field theory states as discussed in paragraph 2: to leading order in z and $\bar{z}$ we have

$$O_{Jm}\approx z^{J+m}\bar{z}^{J-m},\tag{3.11}$$

which can be viewed as representing a particle-hole pair at distances $J+m$ and $J-m$ respectively above and below the fermi surface. The total energy of this state is given by $(J+m)-1/2+(J-m)-1/2=2J-1$, while the momentum is $J+m-(J-m)=2m$. This is consistent with our picture of taking the liouville direction as time.

In this representation, we can now interpret the candidate for the black hole operator, namely

$$\bar{O}_{0,0}\approx O_{N-1,0}=(\bar{\partial}+\frac{1}{2}z)^{N-1}\bar{z}^{N-1}\approx z^{N-1}\bar{z}^{N-1},$$

as the state with maximum possible energy $2J-1$ that can be excited in the system, describing a hole at depth N below the fermi surface, combined with a particle at the distance N above the fermi surface.

It is instructive to compare our singular states to representation given by Witten [14]. A connection is seen as follows: A natural definition of integration over a q-commuting variable z satisfying $z^N=1$ is given by [12] $\int dz\, z^{N-1}=1$. On this space

the delta function is therefore given by $\delta(z) = z^{N-1}$. Now, the negatively dressed states are, from (3.11)

$$
\begin{aligned}
\bar{O}_{Jm} &= O_{N-1-J,m} \\
&\approx z^{N-1-J+m}\bar{z}^{N-1-J-m} \\
&\approx \partial^{J-m}\bar{\partial}^{J+m}\delta(z,\bar{z}),
\end{aligned}
\tag{2.2}
$$

which gives an expression similar to that obtained by Witten.

In summary, we have shown in this section how the phase space of a q-oscillator accomodates both the positively dressed and the negatively dressed states in two-dimensional string theory. We have shown that the algebraic structure, i.e., their commutators, is precisely reproduced, with the positive states closing a W_∞ algebra while the commutators of negative states are seen to vanish. Consequently, this q-oscillator picture provides a framework in which the interaction of all these degrees of freedom can be studied. We emphasize that the negatively dressed states were accomodated not through adding extra degrees of freedom but based on our interpretation that they correspond to states deep below the fermi surface. The use of q-oscillators allowed us to exhibit these states explicitly.

4. Second-quantized representation

As a first step towards understanding the theory at the second-quantized level, we shall now exhibit a second quantized version in terms of free fermions of the algebra (3.5) associated to the hole states only. We shall leave to future work the discussion of combining particles and holes in the full theory, i.e., second quantizing (3.10).

In this spirit, we are therefore motivated to search for a second quantized q-deformed W-algebra in terms of N fermi modes and their conjugates, representing hole creation and annihilation operators. In the following, we will use the q-calculus as described in [12].

Let z and ∂ be q-holomorphic coordinates as defined in (3.8), with $q^{2N} = 1$. Take N femionic modes ψ_{-n}, $n = 0, ..., N-1$, and their conjugates, satisfying the usual anticommutation relations $\{\psi_{-m}, \psi_n^\dagger\} = \delta_{m+n}$. We now define q-holomorphic fields by the expansions

$$
\begin{aligned}
\psi(z) &= \sum_{n=0}^{N-1} \psi_{-n} z^n \\
\psi^\dagger(z) &= \sum_{n=0}^{N-1} z^{N-1-n}\psi_n^\dagger,
\end{aligned}
\tag{4.1}
$$

Here the modes of ψ and $\psi^\dagger$ are taken to commute with z. The reason for the

peculiar power of z in the expression for $\psi^\dagger$ comes from the definition of the dz-integral as $\int dz\, z^{N-1} = 1$ (see [12]) so that z^{N-1} plays a role similar to that of z^{-1} in the commuting theory. We will discuss the generators in terms the q-position space representation below, but first let us try to guess their form in the momentum representation.

Now, in a second-quantized representation of the q-deformed W-algebra, the coproduct becomes important, as it allows us to consistently define the action of the generators on multi-particle states without spoiling the q-commutation relations. In our case, we can check that the following coproduct preserves the q-commutators:

$$\Delta((z^{n+k}\partial^k) = z^{n+k}\partial^k \otimes q^{2k\hat{N}} + q^{2k\hat{N}} \otimes z^{n+k}\partial^k.$$

This coproduct is different from the one described in [11]. However because of its symmetry, it can now easily be extended to multi-particle states, motivating the following expression for the second-quantized generator:

$$W_n^{(k)} = \sum_p \psi^\dagger_{p+n}\, [p]\, [p-1]...[p-k+1]\, q^{2kN_\psi}\, \psi_{-p}, \tag{4.2}$$

where $N_\psi = \sum n\, \psi^\dagger_n\, \psi_{-n}$, and where the ranges of summation are taken such that p and $p+n$ fall inside $[0, ..., N-1]$. Here we remind the reader that $[x] \equiv (q^{2x} - 1)/(q^2-1)$.

To see that this works, it is straightforward to check the crucial fact that, because of the presence of the q^{2kN_ψ}, the quartic terms in the fermion oscillators cancel in the q-bracket (adjusted here for the notation in terms of k and n)

$$q^{(nk'-n'k)}\, W_n^{(k)} W_{n'}^{(k')} - q^{-(nk'-n'k)}\, W_{n'}^{(k')} W_n^{(k)},$$

which therefore reduces to

$$\sum_p \psi^\dagger_{p+n+n'}\, (p^{(nk'-n'k)}\, [p+n']...[p+n'-k+1]\, [p]...[p-k'+1]$$
$$-p^{-(nk'-n'k)}\, [p+n]...[p+n-k'+1]\, [p]...[p-k+1])\, q^{2(k+k')N_\psi}\, \psi_{-p}.$$

Now the term in the inner brackets is just $[z^{n+k}\partial^k, z^{n'+k'}\partial^{k'}]_q$ applied to the basis element z^p, so that the correspondence with the first-quantized algebra follows.

It is important to realize that the generators (4.2) of the second-quantized q-W-algebra are expressed in terms of ordinary fermionic operators with no mention of any exotically commuting quantities. However, these generators satisfy a simple algebra with respect to q-brackets instead of ordinary commutators. Even so, we stress that the above expressions do not contain any degrees of freedom other than those already present in the ordinary fermion theory.

It is possible to formulate the theory also in q-position space in a very nice form. Taking the fields as a function of z as in (4.1), we define

$$W_n^{(k)} \equiv \int dz\, \psi^\dagger(z) z^{(n+k)} \partial^k \psi,$$

with z and ∂ as in (3.8). This leads to the expression (4.2) if we define the z-integral as $\int dz z^{N-1} = 1$ (see Floratos), and if in addition we take the following braiding relation between ψ and ∂:

$$\partial \psi_{-n} = q^{-2n} \psi_{-n} \partial, \tag{4.3}$$

(which can always be compensated by the redefinition $\partial \to q^{2N_\psi} \partial$).

Now, to calculate the q-bracket of two such expressions in z-space, take two "time slices" indexed by z and w and following [15] impose the following braiding relations:

$$zw = wz, \quad z\partial_w = q^{-2} \partial_w z, \quad w\partial_z = q^{-2} \partial_z w, \quad w\, dz = dz\, w, \quad \partial_w dz = q^2 dz\, \partial_w, ... \tag{4.4}$$

which will give us consistent associative differentiation and integration rules. Note that these braiding relations are different from the ones between z and $\bar{z}$ in (3.9) (which formally can be considered as conjugate variables on the same "time slice"). Now, being careful with powers of q picked up when commuting integrations, etc., a straightforward but tedious calculation gives indeed

$$[W_n^{(k)}, W_{n'}^{(k')}]_q = \int dz\, \psi^\dagger(z) \left([z^{n+k} \partial^k, z^{n'+k'} \partial^{k'} k] \right) \psi(z),$$

in other words, the quartic terms cancel due to the braiding relations (4.3) and (4.4) and we are left with an isomorphism between the field theoretic and the z, ∂_z representation. The proof of the above proceeds analogous to the commuting case by noting that according to the z-integration we have

$$\delta(z-w) = \sum_{p=0}^{N-1} z^p w^{N-1-p} = \{\psi(z), \psi^\dagger(w)\}.$$

We close with a remark on dynamics: One might decide to take the above formulation of the theory for finite N as fermions defined on a quantum holomorphic space as fundamental, and define the dynamics accordingly. For related discussions of the natural appearance of quantum groups in string theory, see [16].

In this spirit, let us indicate in broad terms how the calculation of in-out reflection coefficients for the fermions may be approached in such a theory. One way to derive these coefficients is, following [17], to write the fermions in a holomorphic

basis in terms of powers of $z = x + ip$ (for simplicity we now only consider the right side up harmonic oscillator), in which case the in-out transformation is obtained by reexpressing the in-field in terms of the canonically conjugate coordinates $\bar{z} = x - ip$. The in-out bogoliubov transformation thus becomes a fourier transform.

We have described a formulation in terms of fermions on a noncommuting q-holomorphic space. If q-space is fundamental, we would expect the reflection coefficients to be given by a q-fourier transform. A possible definition of such a transform is given by

$$F(z) \equiv \int d\bar{z}\, e_q^{z\bar{z}} f(\bar{z}),$$

where the q-exponential is defined as [12]

$$e_q^{az\bar{z}} \equiv \sum_{p=0}^{2N-1} a^p \frac{(\bar{z}z)^p}{[p]!},$$

where one can check that the inverse transform is given by

$$f(\bar{z}) = \int dz \equiv e_{q^{-1}}^{-q^2\bar{z}z} F(z),$$

which is reasonable if one takes into account that $q \to q^{-1}$ if we interchange z with $\bar{\partial}$ and $\bar{z}$ with ∂ in the q-commutation relations.

Now a careful evaluation of the q-fourier transform of a field $\psi_{-k}\bar{z}^k$ commuting with z and $\bar{z}$ gives

$$\int d\bar{z}\, e_{q^{-1}}^{-q^2 z\bar{z}}\, \bar{z}^k\, \psi_{-k} = \frac{1}{[N-k-1]!} z^{2N-1-k} \psi_{-k},$$

while for its conjugate $\psi_k^\dagger\, \bar{z}^{2N-1-k}$ we should define it as

$$\psi_k^\dagger\, \bar{z}^{N-1-k}\, e_q^{\bar{z}z}\, d\bar{z} \int_{\leftarrow} = \frac{q^{k(k+1)}}{[k]!}\, \psi_k^\dagger\, z^k,$$

where the right to left integral is defined in the obvious way. But we can show that $q^{k(k+1)}/[k]! = [2N-1-k]!$ up to a k-independent factor, in other words, ψ and $\psi^\dagger$ transform dually. This is needed for one obvious consistency condition, namely that the operator $N_\psi = \sum n\psi_n^\dagger \psi_{-n}$ be invariant under an in-out transformation, to be satisfied.

The q-reflection coefficients are seen to be simple generalizations of what one would obtain in the harmonic oscillator. This analysis would correspond in the $q \to 1$ limit to the case of the right side up harmonic oscillator potential. It would be interesting to carefully study the non-analytically-continued case. These and related questions are left to future work.

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